

Assignment 1: Key Concepts and Problem Map

Chapter 1 — linear systems, pivots, transformations

Jingren Wang

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The problem map: 17 questions, five ideas

Concept cluster	Problems	The question it asks	Deciding test
A. Exactly one solution	Q1 Q2 Q4 Q5 Q9	Does elimination pin down every variable?	Pivot in every variable column, no contradictory row
B. Consistency & free variables	Q3 Q6 Q7 Q8 Q11	Is there <i>any</i> solution? More than one?	No row $[0 \cdots 0 \mid c \neq 0]$; free variables give parameters
C. Pivots & rank	Q10 (feeds B, E)	How many independent constraints?	$\text{rank}(A) = \text{number of pivots in the RREF}$
D. Vector & matrix equations	Q12 Q13 Q14	What does $A\mathbf{x}$ mean, and how do the pieces combine?	Component-wise equality; linearity; $A^{-1}\mathbf{b}$ when $\det \neq 0$
E. Onto & one-to-one	Q15 Q16 Q17	Is every output hit? At most one pre-image?	Pivots in every row (onto) / every column (one-to-one)

- Within a cluster the problems are the *same question with different arithmetic*.

A · Elimination pins every variable (Q1, Q2, Q4, Q5, Q9)

Shared shape

Row reduce $[A \mid \mathbf{b}]$: a pivot in *every variable column* and no contradictory row $\Rightarrow$ exactly one solution.

Problem	Shape	Unique answer
Q1	3 equations, exact fractions	$x = -402/265, y = -221/530, z = 71/106$
Q2	3×3	$(x, y, z) = (1, 2, 1)$
Q4	3×3	$(x, y, z) = (5, -5, 5)$
Q5	3×3	$(x, y, z) = (7/5, -3/5, 0)$
Q9	4 equations, 4 unknowns	$(x_1, x_2, x_3, x_4) = (-3, 5, 6, -1)$

A · The recipe in one picture (Q2)

$$\underbrace{\left[\begin{array}{ccc|c} -2 & -3 & 1 & -7 \\ 5 & -4 & -6 & -9 \\ -3 & 5 & -2 & 5 \end{array} \right]}_{\text{start}} \xrightarrow[\text{clear below}]{\text{scale } R_1} \underbrace{\left[\begin{array}{ccc|c} 1 & 3/2 & -1/2 & 7/2 \\ 0 & -23/2 & -7/2 & -53/2 \\ 0 & 19/2 & -7/2 & 31/2 \end{array} \right]}_{\text{first column cleared}} \xrightarrow[\text{clear above}]{\text{repeat}} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]}_{\text{RREF}}$$

Verify before you commit

$$-2(1) - 3(2) + 1(1) = -7, \quad 5(1) - 4(2) - 6(1) = -9, \quad -3(1) + 5(2) - 2(1) = 5 \checkmark$$

- Q1, Q4, Q5, Q9 are the identical recipe — only the arithmetic differs. Q9 shows the same mechanics with four unknowns.

B · One forbidden row decides everything (Q6)

The test

$Ax = b$ is **consistent** $\iff$ the RREF of $[A \mid b]$ contains *no* row $[0 \cdots 0 \mid c]$ with $c \neq 0$ — a row that asserts the false equation $0 = c$.

Q6 · System 1

$$\begin{cases} 3x + 4y + 5z = 1 \\ 3x - 4y + 5z = 1 \\ x + y + z = 1 \end{cases}$$

↓ row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

U — pivots in every column, no forbidden row $\Rightarrow$ unique solution
(2, 0, -1).

Q6 · System 2

$$\begin{cases} 3x + 4y + 5z = 1 \\ x + 2y + 3z = 1 \\ 4x + 6y + 8z = 2 \end{cases}$$

↓ row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

I — no forbidden row, but rank $2 < 3$
leaves z free $\Rightarrow$ infinitely many.
(Row 3 = row 1 + row 2 exactly.)

Q6 · System 3

$$\begin{cases} 3x + 4y + 5z = 1 \\ x + 2y + 3z = 1 \\ 4x + 6y + 8z = 4 \end{cases}$$

↓ row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

N — the forbidden row says $0 = 1$:
inconsistent. (Same left-hand side as
System 2 — the right side alone
changed: $2 \rightarrow 4$.)

B · The test in action: hunt the forbidden row (Q8, Q11)

Q11 — choose k so that the system has no solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 1 & 2 & -3 & 3 \\ 3 & 8 & k & 12 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 3R_1]{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & 1 & -7 & 1 \\ 0 & 5 & k-12 & 6 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & k+23 & 1 \end{array} \right]$$

Read the last row

$0 \cdot x + 0 \cdot y + (k + 23)z = 1$: if $k + 23 \neq 0$ the system has a (unique) solution; if $k = -23$ the row reads $0 = 1$ — the forbidden row — so **no solution exactly at** $k = -23$.

Q8 — A contradictory example:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$x_1 = 0$ together with $x_1 = 1$ (plus $x_2 = 0$): the RREF ends in $0 = 1$. Three equations in five variables and already inconsistent — more variables never rescues a contradiction.

B · Free variables become parameters (Q3, Q7)

Q3 — rank 2, three unknowns:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 10 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] \implies \begin{array}{l} x = 10 - 5z \\ y = 4 - 2z \\ z = z \end{array}$$

One free variable $\Rightarrow$ a *line* of solutions:
infinitely many.

Q7 — rank 2, four unknowns:

$$\mathbf{x} = \begin{bmatrix} 7 \\ 8 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -11 \\ -14 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ -11 \\ 0 \\ 1 \end{bmatrix}$$

Particular solution + homogeneous directions;
two free variables $\Rightarrow$ a plane of solutions.

C · Count pivots once, use the number everywhere (Q10)

$$\begin{bmatrix} 1 & 0 & -6 & -8 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) **2** pivots

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) **1** pivot

$$\begin{bmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

(c) **3** pivots

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -10 \end{bmatrix}$$

(d) **4** pivots

Why this one number buys so much

$\text{rank}(A)$ = number of pivots — and this single count decides four different assignment questions: **Q14** (unique answers), **Q11** (inconsistency), **Q15** (onto), **Q17** (one-to-one).

C · The count in action I: uniqueness & consistency (Q14, Q11)

Q14 — full rank $\iff \det \neq 0$: unique answers

$A = \begin{bmatrix} 3 & 8 \\ -2 & -3 \end{bmatrix}$ row reduces to I : two pivots, full rank, so $\det(A) = 7 \neq 0$ and $A^{-1} = \frac{1}{7} \begin{bmatrix} -3 & -8 \\ 2 & 3 \end{bmatrix}$; every $A\mathbf{x} = \mathbf{b}$ has exactly one solution: (a) $(0, 0) \rightarrow (0, 0)$ trivial; (b) $(18, -5) \rightarrow (-2, 3)$; (c) $(-9, -1) \rightarrow (5, -3)$.

Q11 — one extra pivot in $[A \mid \mathbf{b}] = \text{no solution}$

Elimination leaves $[0 \ 0 \ k+23 \mid 1]$. For $k \neq -23$: three pivots, so $\text{rank}(A) = \text{rank}[A \mid \mathbf{b}] = 3 \Rightarrow$ unique solution. For $k = -23$: the row collapses to $[0 \ 0 \mid 1]$ — an extra pivot in the rightmost column: $\text{rank}[A \mid \mathbf{b}] = 3 > \text{rank}(A) = 2 \Rightarrow 0 = 1$, no solution. **Answer:** $k = -23$.

Takeaway (square matrices)

n pivots in an $n \times n$ matrix $\iff \det \neq 0 \iff$ exactly one solution for every $\mathbf{b}$; an extra pivot in the rightmost column of $[A \mid \mathbf{b}]$ is a contradiction.

C · The count in action II: onto & one-to-one (Q15, Q17)

Q15 — onto $\iff$ pivot in every *row*
(rank = s)

$T : \mathbb{R}^r \rightarrow \mathbb{R}^s$, A of size $s \times r$. Quiz options here:
A = not onto, **B** = onto, **C** = not enough information to tell. Two rows without pivots $\rightarrow$ **A** (rank = $s-2 < s$); a zero row $\rightarrow$ **A** (rank $< s$); a pivot in every *column* (that says rank = r , a one-to-one fact) $\rightarrow$ **C**; every row has a pivot $\rightarrow$ **B** (rank = s).

Q17 — one-to-one $\iff$ pivot in every *column* (rank = r)

Quiz options here: **A** = not one-to-one, **B** = one-to-one, **C** = not enough information to tell. Always rank(A) $\leq \min(r, s)$. If $r > s$: rank $\leq s < r$, the columns are *forced* to be dependent $\rightarrow$ **A**; $r < s$: **C**; $r = s$: **C** — the rank decides.

Takeaway (the row/column mirror)

Onto watches the *rows* — is every target reached? **One-to-one** watches the *columns* — do inputs collide? The same rank, counted in two directions.

D · A vector equation is several scalar equations (Q12)

$$5 \begin{bmatrix} 1 \\ a \\ -7 \end{bmatrix} + 6 \begin{bmatrix} -6 \\ 1 \\ b \end{bmatrix} = \begin{bmatrix} c \\ 6 \\ 6 \end{bmatrix} \iff \begin{aligned} 5(1) + 6(-6) &= c \\ 5a + 6(1) &= 6 \\ 5(-7) + 6b &= 6 \end{aligned}$$

Read off, row by row

$$c = -31, \quad a = 0, \quad b = 41/6.$$

- Equality of vectors means equality *component by component*: each row isolates one unknown, no elimination required.
- The scalars 5 and 6 multiply *every* entry of their vector — the most common slip is applying them to one component only.

D · Linearity and invertibility do the work (Q13, Q14)

Q13 — linearity gives the answer without any matrix entries

$A\mathbf{x} = -2\vec{a} - 4\vec{b} = -2(A\mathbf{u}) - 4(A\mathbf{v}) = A(-2\mathbf{u} - 4\mathbf{v})$, so $\mathbf{x} = -2\mathbf{u} - 4\mathbf{v} = (-8, -20)$. Linearly independent columns $\Rightarrow$ this candidate is the *only* solution.

Q14 — one matrix, three right-hand sides

$A = \begin{bmatrix} 3 & 8 \\ -2 & -3 \end{bmatrix}$, $\det(A) = 7 \neq 0$, $A^{-1} = \frac{1}{7} \begin{bmatrix} -3 & -8 \\ 2 & 3 \end{bmatrix} \quad \Rightarrow \quad$ (a) $(0, 0)$, (b) $(-2, 3)$, (c) $(5, -3)$.

- Part (a) is the homogeneous case: $\det \neq 0$ leaves only the *trivial* solution — the very fact that makes T one-to-one in Q17.

E · The vocabulary first (Q16)

Match each concept to its quantifiers

Concept	Definition	Q16 answer
f is onto	every $\mathbf{y}$ has <i>some</i> $\mathbf{x}$: $\forall \mathbf{y} \exists \mathbf{x}, f(\mathbf{x}) = \mathbf{y}$	A
f is one-to-one	every $\mathbf{y}$ has <i>at most one</i> $\mathbf{x}$	C
f is a (total) function	every $\mathbf{x}$ produces <i>a</i> $\mathbf{y}$: $\forall \mathbf{x} \exists \mathbf{y}$	B

- Option D — “every $\mathbf{y}$ has a *unique* $\mathbf{x}$ ” — is onto *and* one-to-one at once: a **bijection**. It is not the definition of any single one of the three.
- Read quantifiers literally: “at least one” (onto) and “at most one” (one-to-one) can hold simultaneously.

One elimination answers four different questions

- ① **Consistent?** no row $[0 \cdots 0 \mid c \neq 0]$ (Q2–Q6, Q8, Q11)
- ② **How many solutions?** count the free variables (Q1, Q3, Q7)
- ③ **Onto?** pivot in every *row*: $\text{rank} = s$ (Q15, Q16)
- ④ **One-to-one?** pivot in every *column*: $\text{rank} = r$ (Q15, Q17)

The engine behind all four

Free variables (unknowns outnumbering pivots) create nontrivial solutions and infinite families; pivot coverage of the rows and of the columns decides onto and one-to-one. One row reduction — four questions answered.

A: Q1 Q2 Q4 Q5 Q9 • B: Q3 Q6 Q7 Q8 Q11 • C: Q10 • D: Q12 Q13 Q14 • E: Q15
Q16 Q17

Answer key (instructor copy, reference only)

- Q1: unique; $x = -402/265$, $y = -221/530$, $z = 71/106$
- Q2: B; $(1, 2, 1)$
- Q3: E; RREF $\begin{bmatrix} 1 & 0 & 5 & 10 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $x = 10 - 5z$, $y = 4 - 2z$
- Q4: B; $(5, -5, 5)$
- Q5: $(7/5, -3/5, 0)$
- Q6: U , I , N
- Q7:
 $[7, 8, 0, 0]^T + s[-11, -14, 1, 0]^T + t[-9, -11, 0, 1]^T$
- Q8: false
- Q9: $(-3, 5, 6, -1)$
- Q10: (a) 2, (b) 1, (c) 3, (d) 4 pivots
- Q11: $k = -23$
- Q12: $a = 0$, $b = 41/6$, $c = -31$
- Q13: $\mathbf{x} = (-8, -20)$
- Q14: (a) $(0, 0)$; (b) $(-2, 3)$; (c) $(5, -3)$
- Q15: A, A, C, B
- Q16: A, C, B
- Q17: A, C, C

Review map I — from the questions back to the class slides (Q1–Q11)

Questions	Topic to review	Class slides (file · pages)
Q1 Q2 Q4 Q5 Q9	Gaussian elimination from the augmented matrix; elementary row operations; the five-step recipe	CH1.1–1.2(1).pdf: pp. 15–19, 20–21, 37–46, 53–54
Q3	echelon vs. reduced echelon form; free variables and parametric descriptions of solution sets	CH1.1–1.2(1).pdf: pp. 27–29, 47–51 · CH 1.5–1.7.pdf: p. 7
Q7	general solution = particular vector + combination of homogeneous solutions; the four-step parametric recipe	CH 1.5–1.7.pdf: pp. 8–13 (esp. p. 13)
Q6 Q8 Q11	consistent vs. inconsistent; the forbidden row $0 = c$; the Existence and Uniqueness Theorem	CH1.1–1.2(1).pdf: pp. 11, 22–25, 52
Q10	pivot positions and pivot columns	CH1.1–1.2(1).pdf: pp. 30–36

Review map II — vectors, matrices, transformations (Q12–Q17)

Questions	Topic to review	Class slides (file · pages)
Q12	column vectors, scalar multiples, sums, linear combinations (the $4\mathbf{u} + (-3)\mathbf{v}$ example is Q12's exact computation)	CH 1.3–1.4.pdf: pp. 6–19 (esp. p. 7)
Q13	$A\mathbf{x}$ as a linear combination of the columns; $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and $A(c\mathbf{u}) = c A\mathbf{u}$; linearity of T	CH 1.3–1.4.pdf: pp. 21, 32 · CH1.8–1.9.pdf: pp. 10, 13
Q14	the matrix equation $A\mathbf{x} = \mathbf{b}$; homogeneous systems and the trivial solution; the 2×2 inverse formula	CH 1.3–1.4.pdf: pp. 21, 27 · CH 1.5–1.7.pdf: pp. 2–4 · CH2.1–2.2–2.3(1).pdf: pp. 20–21, 34–36
Q15 Q16 Q17	linear transformations and standard matrices; onto and one-to-one definitions; the theorems behind the answers (span, trivial solution, pivots in every row / column)	CH1.8–1.9.pdf: pp. 2–3, 10, 15–21 (onto: pp. 15–16, 18–19; one-to-one: pp. 17, 20–21)

Going further (rank, column space, null space): CH2.4–2.5–2.8–2.9.pdf: pp. 27–29, 42–44.

Thank you!

Questions? Comments? Corrections?

Write to me: `jwang929@connect.hkust-gz.edu.cn`