

Determinants in One Example

Chapter 3 review & tutorial — sections 3.1 & 3.2 through a single 3×3 matrix

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One matrix, the whole chapter

Our single example for the entire session:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \det A = -1 \quad (\text{computed on the next slides}).$$

Lecture part	Concept	What we do with A
3.1	cofactor expansion; any row or column; triangular shortcut	expand along row 1, then column 2; row-reduce to triangular
3.2	row operations; invertibility; $\det A^T$; $\det(AB)$	add, swap, scale a row; make rows dependent; transpose; multiply by $\text{diag}(2, 1, 3)$

- One worked matrix, every formula: you always see *numbers you already know* next to the new symbol.

3.1 · Cofactor expansion along row 1

Definition (for $n \geq 2$)

$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$, where the cofactor is $C_{ij} = (-1)^{i+j}M_{ij}$ and the *minor* M_{ij} is the determinant of the submatrix left after deleting row i and column j . The same recipe works from any row or column.

$$\det A = 1 \cdot \underbrace{\det \begin{bmatrix} 5 & 3 \\ 0 & 8 \end{bmatrix}}_{M_{11}=40} - 2 \cdot \underbrace{\det \begin{bmatrix} 2 & 3 \\ 1 & 8 \end{bmatrix}}_{M_{12}=13} + 3 \cdot \underbrace{\det \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}}_{M_{13}=-5} = 40 - 26 - 15 = \boxed{-1}$$

- Signs alternate $+, -, +$ along row 1 — that is $(-1)^{1+j}$.
- Notation: $\det A = |A|$ — vertical bars replace the brackets.

3.1 · Any row or column gives the same value

Expand along *column 2* this time (its zero entry saves one minor):

$$\det A = \underbrace{(-1)^{1+2}}_{-} 2 \cdot \det \begin{bmatrix} 2 & 3 \\ 1 & 8 \end{bmatrix} + \underbrace{(-1)^{2+2}}_{+} 5 \cdot \det \begin{bmatrix} 1 & 3 \\ 1 & 8 \end{bmatrix} + \underbrace{(-1)^{3+2}}_{-} 0 \cdot \det \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = -2(13) + 5(5) - 0 = \boxed{-1}$$

Why this matters

- All $2n$ possible expansions (any of n rows, any of n columns) give the *same* number — -1 here, every time.
- Strategy: expand along the row or column with the most zeros; each zero kills one minor. With no zeros anywhere, the work is the same everywhere — so pick row 1 and go.

3.1 · The triangular shortcut — via row reduction

Row operations (Chapter 2!) turn A into a triangular matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

Theorem (triangular matrices)

det of a triangular matrix = product of its diagonal entries: $1 \cdot 1 \cdot (-1) = -1$ — matching the cofactor expansion exactly.

- No row swaps or scalings were needed, so nothing else adjusts the product. This is $\det A =$ product of the *pivots* — the count from Chapter 2 becomes a number: $\text{rank}(A) = 3$, three pivots, $|\det| = 1$.

3.2 · The three row operations, tested on A

Operation	New matrix	det
$R_2 \rightarrow R_2 + R_1$ (add a multiple of a row)	$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 6 \\ 1 & 0 & 8 \end{bmatrix}$	unchanged: $56 - 36 - 21 = -1$
$R_1 \leftrightarrow R_2$ (swap two rows)	$\begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$	sign flips: $32 - 25 - 6 = +1$
$R_1 \rightarrow 2R_1$ (scale a row by k)	$\begin{bmatrix} 2 & 4 & 6 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$	multiplied by k: $2 \cdot (-1) = -2$

- Each right-hand side was expanded along row 1 — same minors as before, recombined.
- These three rules are exactly why $\det = \text{product of pivots}$: only swaps and scalings ever touch the product, additions are free.

3.2 · When the rows depend on each other: $\det = 0$

Replace row 3 by $R_1 + R_2$ (a legitimate row operation — $\det$ should survive):

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 3 & 7 & 6 \end{bmatrix} = 1(30 - 21) - 2(12 - 9) + 3(14 - 15) = 9 - 6 - 3 = \boxed{0}$$

The determinant detects dependence

$R_3 = R_1 + R_2$ makes the rows linearly *dependent*: row reduction now produces a zero row, $\text{rank}(A) = 2 < 3$, so A is **singular** — no inverse, $A\mathbf{x} = \mathbf{b}$ not solvable for every $\mathbf{b}$.

Theorem (invertibility)

A is invertible $\iff \det A \neq 0$. Our original A : $\det A = -1 \neq 0 \Rightarrow$ invertible, three pivots, $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$.

Think from Chapter 1 and 2 (board discussion)

The statement to explain

“The modified A (with $R_3 = R_1 + R_2$) is **singular** — $Ax = \mathbf{b}$ is **not solvable for every $\mathbf{b}$.**”

Can you explain it *in dimension space*?

(open question — we build the picture on the board together)

3.2 · Transpose and products

$$\det A^T = \det A$$

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 3 & 3 & 8 \end{bmatrix}: \text{expand along row 1,}$$

$$1(40) - 2(16) + 1(6 - 15) = 40 - 32 - 9 = -1.$$

Rows become columns, the determinant does not care.

$$\det(AB) = \det A \cdot \det B$$

Take $B = \text{diag}(2, 1, 3)$, so $\det B = 6$:

$$AB = \begin{bmatrix} 2 & 2 & 9 \\ 4 & 5 & 9 \\ 2 & 0 & 24 \end{bmatrix} \implies \det(AB) = -6,$$

computed by expanding along column 2:
 $-2(96 - 18) + 5(48 - 18) = -6$. And indeed
 $\det A \cdot \det B = (-1)(6) = -6$.

- Bonus: AB is A with column 1 scaled by 2 and column 3 by 3 — so this is also the *column*-version of the scaling rule.
- Warning the other way: $\det(A + B) \neq \det A + \det B$ in general.

One matrix, one number: sections 3.1 & 3.2

Concept	Statement	On our A
3.1 cofactor expansion	$\det A = \sum_j a_{ij}(-1)^{i+j} M_{ij}$, any row/column	-1 (row 1 and column 2 agree)
3.1 triangular	$\det =$ product of diagonal entries	$1 \cdot 1 \cdot (-1) = -1$
3.2 row ops	add: free; swap: flip sign; scale: multiply by k	$-1, +1, -2$
3.2 dependence	rows dependent $\iff \det = 0 \iff$ singular	$R_3 = R_1 + R_2$ gave 0
3.2 transpose & product	$\det A^T = \det A$; $\det(AB) = \det A \det B$	-1 ; $(-1)(6) = -6$

- Everything above reused the *same* nine entries of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ — master this one computation and sections 3.1–3.2 are a checklist.

Thank you!

Questions? Comments? Corrections?

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