# Vector Spaces in One Example

## Chapter 4 review & tutorial — 4.1–4.6 through a single $3 \times 4$ matrix

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# One matrix, the whole chapter

## The single example — a $3 \times 4$ matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 5 & 5 \\ -1 & 1 & -1 & -4 \end{bmatrix}$$

| Lecture part | Concept | What we do with $A$ |
|---|---|---|
| 4.1 | vector spaces, subspaces, Span | the plane $H$ spanned by columns 1, 2 |
| 4.2 | Nul $A$, Col $A$, membership tests | spanning set for Nul $A$; is $\mathbf{b}$ in Col $A$? both answers |
| 4.3 | independence, bases, pivot columns | bases for Col $A$, Row $A$, Nul $A$; drop dependent columns |
| 4.4 | coordinates $[\mathbf{y}]_{\mathcal{B}}$, change of basis | $[\mathbf{b}_1]_{\mathcal{B}} = (1, 1)$; change-of-basis matrix $P$ |
| 4.5 | every basis has the same size | both bases of the plane: 2 vectors $\Rightarrow \dim = 2$ |
| 4.6 | rank, rank of $A^\top$ | $\operatorname{rank}(A) = \operatorname{rank}(A^\top) = 2$ |
| Rank Theorem | $\operatorname{rank} + \dim \operatorname{Nul} = n$ | $2 + 2 = 4$ |

# 4.1 · A subspace, in and around $A$

## Subspace definition

$H \subseteq V$ is a subspace iff (i) $\mathbf{0} \in H$, (ii) closed under addition, (iii) closed under scalar multiplication. $\mathbb{R}^3$ itself is a vector space; $\{\mathbf{0}\}$ is its smallest subspace.

## $H = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ — the plane

With $\mathbf{a}_1 = (1, 2, -1)$ and $\mathbf{a}_2 = (0, 1, 1)$ (columns 1, 2 of $A$), a generic point of the span is

$$(x, y, z) = s\,\mathbf{a}_1 + t\,\mathbf{a}_2 = (s,\ 2s + t,\ -s + t),$$

$$\text{so } x = s,\ y = 2s + t,\ z = -s + t.$$

Eliminating the parameters: $t = y - 2x$ and $z = y - 3x$ — i.e. the plane $3x - y + z = 0$ with normal $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = (3, -1, 1)$.

## Subspace checks

**(i)** $s = t = 0$ gives $\mathbf{0}$ ✓ **(ii)**
$s\mathbf{a}_1 + t\mathbf{a}_2 + s'\mathbf{a}_1 + t'\mathbf{a}_2 = (s{+}s')\mathbf{a}_1 + (t{+}t')\mathbf{a}_2$ ✓ **(iii)**
$c(s\mathbf{a}_1 + t\mathbf{a}_2) = (cs)\mathbf{a}_1 + (ct)\mathbf{a}_2$ ✓
All members satisfy $3x - y + z = 0$, and $\mathbf{0}$ does too: $H$ is a **subspace of** $\mathbb{R}^3$.

- $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is always a subspace — that is the lecture's guarantee we are using here.

# 4.2 · Nul $A$: compute a spanning set, step by step

**Definition:** $\operatorname{Nul} A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$, a subspace of $\mathbb{R}^4$ (for our $3 \times 4$ $A$). Solve $A\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 2 & 1 & 5 & 5 & 0 \\ -1 & 1 & -1 & -4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Free variables $x_3 = s$, $x_4 = t$; back-substitute $x_1 = -2s - 3t$, $x_2 = -s + t$:

$$\mathbf{x} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0} \checkmark \qquad A \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0} \checkmark$$

- $\operatorname{Nul} A$ is a *plane inside* $\mathbb{R}^4$ (two free variables), and $\{(-2,-1,1,0), (-3,1,0,1)\}$ spans it — an explicit description.

# 4.2 · Col $A$: is $\mathbf{b}$ in it? Both answers

**Definition:** $\operatorname{Col} A = \operatorname{Span}\{\text{columns of } A\}$, a subspace of $\mathbb{R}^3$. Membership test: $\mathbf{b} \in \operatorname{Col} A \iff A\mathbf{x} = \mathbf{b}$ is consistent.

## YES — $\mathbf{b}_1 = (1, 3, 0)$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 2 & 1 & 5 & 5 & 3 \\ -1 & 1 & -1 & -4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Consistent ✓: $\mathbf{x} = (1, 1, 0, 0)$ works — $A(1, 1, 0, 0) = (1, 3, 0)$ ✓

## NO — $\mathbf{b}_2 = (1, 0, 0)$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 2 & 1 & 5 & 5 & 0 \\ -1 & 1 & -1 & -4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

$0 = 3$: **inconsistent**, $\mathbf{b}_2 \notin \operatorname{Col} A$.

- Geometric reason: $\operatorname{Col} A$ is the plane $3x - y + z = 0$ from the 4.1 slide; $\mathbf{b}_1$ lies on it, $\mathbf{b}_2 = (1, 0, 0)$ does not ($3 \cdot 1 - 0 + 0 = 3 \neq 0$).

# 4.3 · Bases for Col $A$: drop the dependent columns

**Dependency, read off the columns:** $\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{a}_4 = 3\mathbf{a}_1 - \mathbf{a}_2$ (both verified on slide 2's matrix: $2(1,2,-1) + (0,1,1) = (2,5,-1)$ ✓, $3(1,2,-1) - (0,1,1) = (3,5,-4)$ ✓).

## Spanning Set Theorem

A dependent vector can be removed from a spanning set without changing the span. Applying it twice to $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ leaves $\{\mathbf{a}_1, \mathbf{a}_2\}$: independent (not multiples), still spanning Col $A \Rightarrow$ a **basis**.

## Pivot Columns Theorem — same answer, systematically

The RREF (next slide computes it) has pivots in columns 1, 2; the pivot columns of the *original* $A$ form a basis for Col $A$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \dim \operatorname{Col} A = \operatorname{rank}(A) = 2 \ (< 3 : \operatorname{Col} A \neq \mathbb{R}^3, \ T \text{ not onto}).$$

# 4.3 · Bases for Row $A$ and Nul $A$, step by step

**Row reduce** (two additions and one more — same ops as the Nul slide):

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 5 & 5 \\ -1 & 1 & -1 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Row $A$ — nonzero rows of the echelon form

Rows of any row-equivalent echelon matrix span the same row space, and nonzero echelon rows are independent:

$$\text{basis of } \operatorname{Row} A = \{(1,0,2,3),\ (0,1,1,-1)\}.$$

Original rows are combinations: $2\,r_1 + r_2 = (2,1,5,5)$ ✓, $-r_1 + r_2 = (-1,1,-1,-4)$ ✓

- Rank Theorem: $\operatorname{rank} 2 + \dim \operatorname{Nul} 2 = 4 = n$

## Nul $A$ — needs the RREF (free variables)

Already reduced: $x_1 + 2x_3 + 3x_4 = 0$, $x_2 + x_3 - x_4 = 0$, $x_3, x_4$ free:

$$\text{basis of } \operatorname{Nul} A = \{(-2,-1,1,0),\ (-3,1,0,1)\}.$$

# 4.4 · Coordinates: $[\mathbf{y}]_{\mathcal{B}}$ for the plane

## Theorem 8 — a basis gives every vector unique coordinates

For the basis $\mathcal{B} = \{\mathbf{a}_1, \mathbf{a}_2\}$ of the plane, $\mathbf{y} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2$ has exactly one solution $(c_1, c_2)$: the **coordinate mapping** $[\mathbf{y}]_{\mathcal{B}} = (c_1, c_2)^\top$ is an isomorphism $H \to \mathbb{R}^2$.

## Compute $[\mathbf{b}_1]_{\mathcal{B}}$ — solve $[\,\mathbf{a}_1\ \mathbf{a}_2 \mid \mathbf{b}_1\,]$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$[\mathbf{b}_1]_{\mathcal{B}} = (1, 1)$: $\mathbf{b}_1 = \mathbf{a}_1 + \mathbf{a}_2$ ✓

## Dependencies re-read as coordinates

$[\mathbf{a}_3]_{\mathcal{B}} = (2, 1)$ $(\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_2)$, $\qquad [\mathbf{a}_4]_{\mathcal{B}} = (3, -1)$ $(\mathbf{a}_4 = 3\mathbf{a}_1 - \mathbf{a}_2)$

# 4.4 · Change of coordinates: a second basis

$\mathcal{C} = \{\mathbf{a}_1, \mathbf{a}_4\}$ also spans the plane ($\mathbf{a}_2 = 3\mathbf{a}_1 - \mathbf{a}_4$, verified).

## Change-of-coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$

Columns of $P$: the $\mathcal{C}$-coordinates of the $\mathcal{B}$-vectors — $[\mathbf{a}_1]_{\mathcal{C}} = (1, 0)$, $[\mathbf{a}_2]_{\mathcal{C}} = (3, -1)$:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}, \qquad [\mathbf{b}_1]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Check: $4\mathbf{a}_1 - \mathbf{a}_4 = (1, 3, 0) = \mathbf{b}_1$ ✓

## Standard-basis special case

With $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the change-of-coordinates matrix is simply the matrix of the basis vectors itself, $P_{\mathcal{E}\leftarrow\mathcal{B}} = A$ — the columns ARE the standard coordinates.

# 4.5 & 4.6 · Dimension and Rank

## 4.5 — every basis of a space has the same size: that size is the dimension

Both bases of the plane ($\mathcal{B}$ and $\mathcal{C}$) have 2 vectors: $\dim H = 2$. Other dimensions: $\dim \operatorname{Nul} A = 2$, $\dim \operatorname{Row} A = 2$, $\dim \mathbb{R}^3 = 3$ — so the plane is a *proper* subspace ($2 < 3$).

## 4.6 — Rank Theorem

$\operatorname{rank}(A) = \dim \operatorname{Col} A = 2$ (pivot columns), $\dim \operatorname{Nul} A = 2$ (free variables), and

$$\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = 2 + 2 = 4 = n.\checkmark$$

## 4.6 — rank of the transpose

$\operatorname{rank}(A^\top) = \operatorname{rank}(A) = 2$: the RREF of $A^\top$ (4 rows) again carries 2 pivots — row operations preserve the row space, so the row rank and column rank coincide.

- Big picture for our $3 \times 4$ $A$: rank $2 < 3$ rows $\Rightarrow$ $T(\mathbf{x}) = A\mathbf{x}$ is **not onto** $\mathbb{R}^3$; nullity $2 > 0 \Rightarrow$ **not one-to-one**.

# When is $T$ onto, when one-to-one?

## The two tests for $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$ ($A$ is $m \times n$)

**ONTO** — every output reached: $\text{rank}(A) = m$ (pivot in every *row*); $A\mathbf{x} = \mathbf{b}$ is consistent for *all* $\mathbf{b} \in \mathbb{R}^m$.
**ONE-TO-ONE** — inputs never collide: $\text{rank}(A) = n$ (pivot in every *column*); $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

| Shape of $A$ ($m \times n$) | Onto? | One-to-one? |
|---|---|---|
| wide: $n > m$ | possible, if rank $= m$ | **never**: rank $\leq m < n$ (free variables always) |
| tall: $n < m$ | **never**: rank $\leq n < m$ | possible, if rank $= n$ |
| square: $n = m$ | onto $\iff$ one-to-one $\iff$ invertible (rank $= n \iff \det \neq 0$) | same |

- Our $A$ is wide ($3 \times 4$, rank 2): **not onto**, **not one-to-one**. A tall $4 \times 3$ cousin of rank 3 would be one-to-one (but never onto); a wide $3 \times 4$ of rank 3 would be onto (but never one-to-one).

# One table: the whole chapter

| Concept | On our $A$ | The takeaway |
|---|---|---|
| 4.1 subspace | $H = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$: plane $3x-y+z=0$ in $\mathbb{R}^3$ | a span is always a subspace |
| 4.2 Nul $A$ | all solutions of $A\mathbf{x} = \mathbf{0}$: a plane in $\mathbb{R}^4$, spanned by $(-2,-1,1,0)$, $(-3,1,0,1)$ | lives in $\mathbb{R}^n$; found via RREF free variables |
| 4.2 Col $A$ | span of the columns: the *same* plane in $\mathbb{R}^3$ | lives in $\mathbb{R}^m$; test by solving $A\mathbf{x} = \mathbf{b}$ |
| 4.3 bases | Col: $\{\mathbf{a}_1, \mathbf{a}_2\}$ (pivot columns); Row: 2 echelon rows; Nul: 2 special solutions | pivot columns of the *original* $A$; RREF for Nul |
| 4.4 coordinates | $[\mathbf{b}_1]_{\mathcal{B}} = (1,1)$; change of basis $P_{\mathcal{C}\leftarrow\mathcal{B}} = [[1,3],[0,-1]]$ (self-inverse) | unique weights per basis |
| 4.5/4.6 rank | every basis has the same size; $\text{rank}(A^\top) = \text{rank}(A)$; Rank Theorem: $\text{rank} + \dim \text{Nul} = n$ $(2+2=4)$ | pivots + free variables = unknowns |

# Review map — what to re-study per question style

| If the exam asks ... | Review this slide |
|---|---|
| is $H$ a subspace? / is $\mathbf{w}$ in Span$\{\dots\}$? | 4.1 slide: the three closure checks; membership = solving |
| find a spanning set for Nul $A$ | Nul $A$ slide: RREF, free variables, special solutions |
| is $\mathbf{b}$ in Col $A$? / find Col $A$ | Col $A$ slide: consistency test; geometry (plane condition) |
| find bases for Col $A$, Row $A$, Nul $A$ | the two 4.3 slides: pivot columns, echelon rows, special solutions |
| state the Rank Theorem / compute rank, dim Nul | Rank line of the summary: rank $+$ dim Nul $= n = 4$ here |
| find coordinates $[\mathbf{y}]_{\mathcal{B}}$ / change of basis | 4.4 slides: unique weights; $P_{\mathcal{C}\leftarrow\mathcal{B}}$ (self-inverse here) |
| state a dimension or rank fact (e.g. rank of $A^{\top}$) | 4.5 & 4.6 slides: summary table, last rows |

# Thank you!

Questions? Comments? Corrections?

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