# Application and Implementation of Univariate Kalman Filter

## Lab M4 – Task 4.3

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MICS6002K – Probabilistic Methods for Time Series Analysis

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# Outline

# Problem Statement



## Goal

Apply and implement a univariate Kalman Filter to track a train moving back and forth on a 1D track under noisy sensor measurements.

**Key Challenges:**

- Sensor measurements are corrupted by noise ($R = \sigma^2_{\text{sensor}}$)
- The train reverses direction at track boundaries (nonlinear dynamics)
- The KF assumes a constant velocity model (linear prediction)
- We must evaluate how initialization, noise levels, and process variance affect performance

**Questions to Address:**

1. How to implement KF with and without the Kalman gain formulation?
2. How robust is KF to bad initial estimates?
3. How sensitive is KF to measurement noise and process variance?
4. What is the generic KF algorithm?

# Approach Overview

**System Model (Univariate):**

- State $x_t$: train position at time $t$
- Constant velocity assumption: $v = 1.0$ m/step
- Process noise: $Q$ (variance of unmodeled dynamics)
- Measurement noise: $R = \sigma^2_{\text{sensor}}$

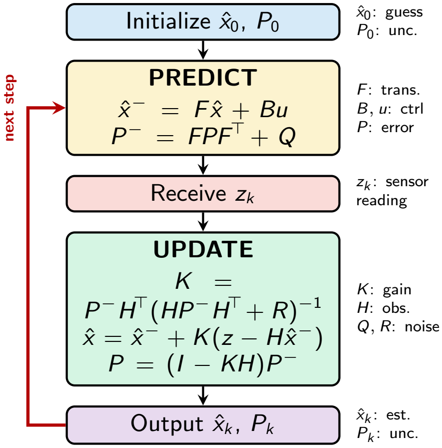
**Two Equivalent Formulations:**

1. **Without Kalman gain:** Direct posterior computation
2. **With Kalman gain:** Using $K$ to blend prediction and measurement

**Implementation Steps:**

1. Generate train trajectory (bouncing ball model)
2. Add Gaussian noise to simulate sensor readings
3. Implement `predict()` and `update()` functions
4. Run experiments with different parameters
5. Analyze robustness and sensitivity

# Generic Kalman Filter Algorithm




**Two-step recursive process:**

1. **Predict** (a priori):

$$\hat{x}^- = F\,\hat{x}_{k-1} + B\,u_k$$
$$P^- = F\,P_{k-1}\,F^\top + Q$$

2. **Update** (a posteriori):

$$K = P^- H^\top (H\,P^- H^\top + R)^{-1}$$
$$\hat{x}_k = \hat{x}^- + K\,(z_k - H\,\hat{x}^-)$$
$$P_k = (I - K\,H)\,P^-$$

**For univariate case:** $F = 1$, $H = 1$, $B = 1$, so all are scalars.

## Direct Bayesian Formulation

Compute the posterior directly from the prior and measurement, without explicitly computing $K$.

**Predict:**

$$\hat{x}^- = \hat{x}_{k-1} + v \cdot \Delta t, \quad P^- = P_{k-1} + Q$$

**Update (no $K$):**

$$\hat{x}_k = \frac{P^- \cdot z_k + R \cdot \hat{x}^-}{P^- + R}, \quad P_k = \frac{P^- \cdot R}{P^- + R}$$

**Intuition:** The posterior is a *precision-weighted average* of the prediction $\hat{x}^-$ and the measurement $z_k$. The weight of each is proportional to the other's *uncertainty* (variance).

# KF With Kalman Gain

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## Kalman Gain Formulation

Introduce $K$ as the optimal blending factor between prediction and measurement.

**Predict:**

$$\hat{x}^- = \hat{x}_{k-1} + v \cdot \Delta t, \quad P^- = P_{k-1} + Q$$

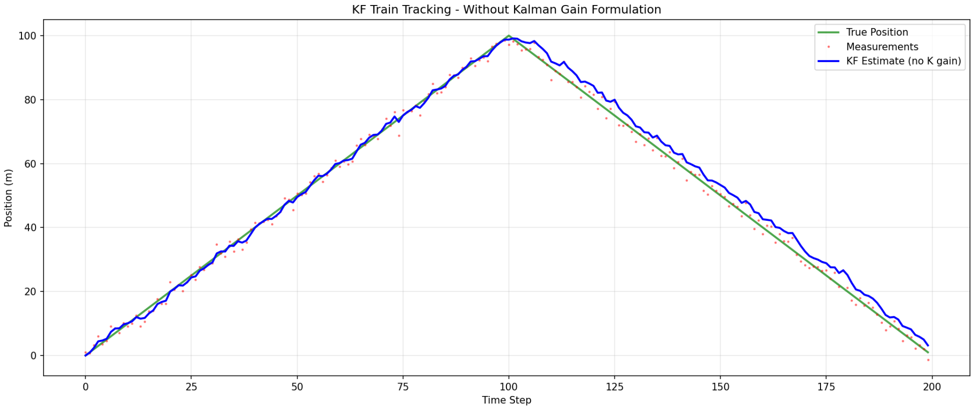
**Update (with $K$):**

$$K = \frac{P^-}{P^- + R}, \quad \hat{x}_k = \hat{x}^- + K\,(z_k - \hat{x}^-), \quad P_k = (1 - K)\,P^-$$

**Kalman Gain Interpretation:**

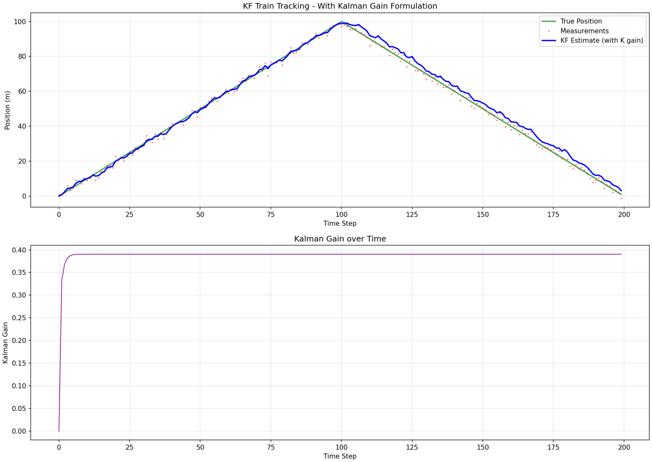
- $K \to 1$ when $P^- \gg R$: trust measurements more
- $K \to 0$ when $R \gg P^-$: trust predictions more
- Both formulations produce **identical results** (verified: max diff $\approx 10^{-14}$)

# Train Tracking Results



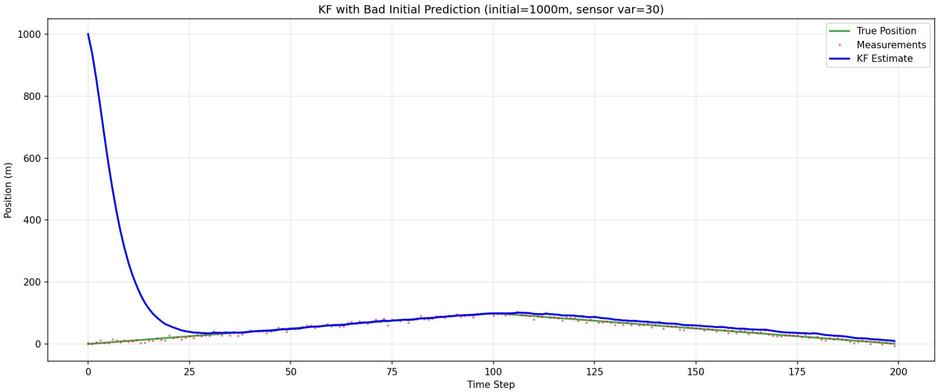

$Q = 1.0$, $R = 4.0$ (sensor $\sigma = 2.0$). KF estimate (blue) smoothly tracks the true position (green) despite noisy measurements (red).

# Kalman Gain Over Time



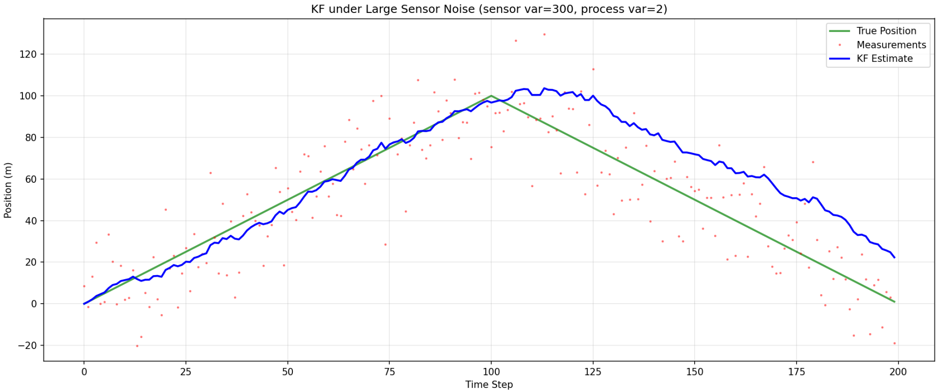

Kalman gain converges to $\approx 0.39$ (Riccati fixed point, $Q = 1.0$, $R = 4.0$).

# Robustness to Bad Initial Estimates



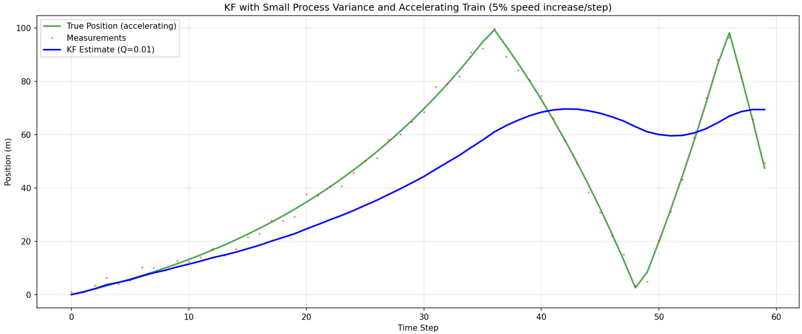

**Initial state:** $\hat{x}_0 = 1000$ (far from true position 0). **Sensor variance:** $R = 30.0$. The filter converges within $\sim$20 steps.

# Sensitivity to Measurement Noise




**Large measurement noise** ($R = 300,\ Q = 2.0$): KF relies more on its process model, producing smoother but lagging estimates. $K$ automatically decreases when $R$ is large.

# Sensitivity to Process Variance




**Very small** $Q = 0.01$ with an accelerating train (5% speed increase/step). The estimate *lags behind* because the filter cannot track changing velocity.

# Robustness Analysis

**Robust to Measurement Noise:**

- $K = \frac{P^-}{P^- + R}$ automatically decreases when $R$ is large
- Filter relies more on its internal model
- Self-regulating mechanism
- No manual threshold tuning needed

**Robust to Initial Errors:**

- Large $P_0 \Rightarrow$ large $K$ initially
- Measurements quickly correct the estimate
- Convergence in $\sim$10–20 steps

**Sensitive to $Q$ and $R$ Tuning:**

- Small $Q$: smooth but sluggish, cannot track acceleration
- Large $Q$: responsive but noisy estimates
- Small $R$: trusts sensor, follows noise
- Large $R$: trusts model, may miss real changes

| Experiment | $Q$ | $R$ | Behavior |
|---|---|---|---|
| Default | 1.0 | 4.0 | Balanced tracking |
| Bad initial | 1.0 | 30.0 | Quick recovery |
| Large noise | 2.0 | 300.0 | Smooth, relies on model |
| Small $Q$ | 0.01 | 4.0 | Lags behind acceleration |

# Limitations of the Constant-Velocity Model

- The train reverses direction at boundaries — this is a **nonlinear** event
- The KF assumes linear dynamics ($\hat{x}^- = \hat{x} + v \cdot \Delta t$)
- At turning points, the prediction overshoots because velocity doesn't change; under acceleration, the estimate lags behind
- **Possible improvements:**
  - Extended Kalman Filter (EKF) for nonlinear dynamics
  - Interacting Multiple Model (IMM) to handle mode switches
  - Adaptive $Q$: increase process noise near boundaries

**Key Takeaway**

The KF is optimal *only* when the model accurately represents the system. Model mismatch is the primary source of error.

# Conclusion



1. **Two equivalent formulations** of the univariate KF produce identical results:
   - Direct Bayesian update (precision-weighted average)
   - Kalman gain formulation (optimal blending factor)
2. **The Kalman gain** $K$ is self-regulating:
   - Adapts to the relative uncertainty of prediction vs. measurement
   - Converges to a steady-state value for constant $Q$, $R$
3. **Robustness:** KF handles large initial errors and noisy measurements gracefully through its built-in variance tracking
4. **Sensitivity:** Performance depends critically on $Q$ and $R$ — proper tuning is essential
5. **Model limitation:** The constant-velocity assumption causes lag when the system accelerates — a nonlinear phenomenon not captured by the linear model

# Thank You

Questions?

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